Hilbert modular forms from orthogonal modular forms on quaternary lattices

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Motivation

Theorem (Lagrange, 1770)

Every positive integer can be written as a sum of four squares.

Example

$$7 = 2^2 + 1^2 + 1^2 + 1^2$$

Question

In how many ways?
$r_4(n) = \#\{\lambda \in \mathbb{Z}^4 : \sum_{i=1}^4 \lambda_i^2 = n\}.$

Example

$$r_4(7) = 4 \cdot 2^4 = 64$$

$$r_4(n) = 8, 24, 32, 8, 48, 96, 64, \ldots$$

;

Sums of four squares

Approach #1

- Write $\theta(q) = \sum_{n=0}^{\infty} r_4(n) q^n$.
- Show that θ belongs to a finite dimensional vector space V.
- Find a basis for V.
- Represent θ in that basis and compare coefficients.

Approach #2

- Use Quaternions - $B\langle i, j\rangle = \left(\frac{-1,-1}{\mathbb{Q}}\right)$, $O = \mathbb{Z}\langle i, j\rangle$.
- Count elements of O with norm n.
- Reduce to count (right) O-ideals with norm n.
- Reduce to prime powers.
- Count separately for $p = 2$ and $p \neq 2$.

Modular curves

- The **upper half plane** is $\mathfrak{H} = \{z \in \mathbb{C} : \Im(z) > 0\}$.
- It admits an action of $\mathrm{GL}_2^+(\mathbb{R})$ by **Möbius transformations**

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \mathfrak{H} \to \mathfrak{H}, \quad z \mapsto \gamma z = \frac{az+b}{cz+d}$$

- For a discrete $\Gamma \leq \mathrm{GL}_2^+(\mathbb{R})$, can form $Y(\Gamma) = \Gamma \backslash \mathfrak{H}$.
- Specific groups Γ of interest

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0 \bmod N \right\}$$

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) : a, d \equiv 1 \bmod N \right\}$$

- Note that $\gamma \mapsto d : \Gamma_0(N) \to (\mathbb{Z}/N\mathbb{Z})^\times$ has kernel $\Gamma_1(N)$.
- Compactify using **cusps**

$$X(\Gamma) = Y(\Gamma) \cup (\Gamma \backslash \mathbb{P}^1(\mathbb{Q})), \quad X_0(N) = X(\Gamma_0(N))$$

Modular forms

- Fact: $X(\Gamma)$ is a compact Riemann surface.

Example

Local coordinate at ∞ for $X_0(N)$ is $q = e^{2\pi i z}$.

- For Γ torsion-free, let $M_2(\Gamma)$ be differentials on $X(\Gamma)$ holomorphic on $Y(\Gamma)$ with at most simple poles at the cusps.

Theorem (Riemann-Roch, 1865)

$$\dim M_2(\Gamma) = g(X(\Gamma)) + \#cusps - 1$$

Let $\pi : \mathfrak{H} \to X(\Gamma)$. For $\omega \in M_2(\Gamma)$ can consider $\pi^*(\omega) = f(z)dz$.

Example

For $X_0(N)$, if near ∞, $\omega = g(q)\ dq$, then $\pi^*(\omega) = 2\pi i\ q\ g(q)\ dz$

Modular forms, cusp forms and characters

Thus $\omega \mapsto f(z)$ identifies $M_2(\Gamma)$ with hol. functions $f : \mathfrak{H} \to \mathbb{C}$, s.t.

$$(cz+d)^{-2} f(\gamma z)\, dz = f(\gamma z)\, d(\gamma z) = f(z)\, dz \quad \forall \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$$

- $M_2(\Gamma)$ is the space of **modular forms** of level Γ (of weight 2).
- Write $S_2(\Gamma) \subseteq M_2(\Gamma)$ for the holomorphic differentials.
- The map $\omega \mapsto f(z)$ identifies $S_2(\Gamma)$ with the functions in $M_2(\Gamma)$ that vanish at the cusps, called **cusp forms**.
- $(\mathbb{Z}/N\mathbb{Z})^\times \simeq \Gamma_0(N)/\Gamma_1(N)$ acts on $M_2(\Gamma_1(N))$ via

$$f(z)d(z) \mapsto f(\gamma_0 z)d(\gamma_0 z)$$

- Write $M_2(N, \chi)$ (resp. $S_2(N, \chi)$) for the χ-isotypic component, so $f \in M_2(N, \chi)$ iff

$$f(\gamma z) = \chi(d)(cz + d)^2 f(z) \quad \forall \gamma \in \Gamma_0(N)$$

$X_0(4)$

Example

We compute that

- $\Gamma_0(4)\backslash\mathbb{P}^1(\mathbb{Q}) = \{0, \frac{1}{2}, \infty\}$,
- $X_0(4) \simeq \mathbb{P}^1(\mathbb{C})$

so $\dim M_2(\Gamma_0(4)) = 2$.
The function $\theta(z) = \sum_{n=0}^{\infty} r_4(n)q^n$ is holomorphic, and

$$\theta(z+1) = \theta(z), \quad \theta\left(\frac{z}{4z+1}\right) = (4z+1)^2\theta(z),$$

hence $\theta \in M_2(\Gamma_0(4))$. [Also invariant under $z \mapsto -\frac{1}{4z}$]

Jacobi's four-square theorem

Theorem (Jacobi, 1834)

$$r_4(n) = 8 \sum_{4 \nmid d | n} d.$$

Proof.

- Construct $E_2(z) = -\frac{1}{24} + \sum_{n=1}^{\infty} \sigma(n) q^n$.
- $v_1 = E_2(z) - 2E_2(2z)$, $v_2 = E_2(2z) - 2E_2(4z) \in M_2(\Gamma_0(4))$.
- From first two terms deduce $\theta(z) = 8v_1 + 16v_2$

$$\sum r_4(n) q^n = 8(E_2(z) - 4E_2(4z)) = 8\left(\sum \sigma(n) q^n - \sum \sigma(n) q^{4n}\right)$$

yields $r_4(n) = \sum_{4 \nmid d | n} d$. □

Representation Numbers

More generally, if $Q(x) = \sum_{i \leq j} a_{ij} x_i x_j$ is a quadratic form with $a_{ij} \in \mathbb{Z}$, we may consider

$$r_Q(n) = \#\{\lambda \in \mathbb{Z}^k : Q(\lambda) = n\}$$

and the function

$$\theta_Q(z) = \sum_{n=0}^{\infty} r_Q(n) q^n = \sum_{\lambda \in \mathbb{Z}^k} q^{Q(\lambda)}$$

is again a modular form.

Quadratic forms and Lattices

Let $Q : V \to \mathbb{Q}$ be a positive definite quadratic space with $\dim_{\mathbb{Q}} V = k$ with associated bilinear form

$$T(x, y) := Q(x + y) - Q(x) - Q(y).$$

Let $\Lambda \subseteq V$ be an **even integral** lattice, so that $Q(\Lambda) \subseteq \mathbb{Z}$.

Define $\Delta = \mathrm{disc}(\Lambda) = \det T \in \mathbb{Z}$, and let $\Delta^* = \begin{cases} \Delta & 2 \nmid k \\ (-1)^{k/2}\Delta & 2 \mid k \end{cases}$

Given a lattice, we may construct associated **theta series**

$$\theta_\Lambda(z) = \theta_\Lambda^{(1)}(z) = \sum_{\lambda \in \Lambda} q^{Q(\lambda)}, \quad q = e^{2\pi i z}$$

The **level** of Λ is the smallest N such that $NQ(\Lambda^\sharp) \subseteq \mathbb{Z}$.
Then $\theta_\Lambda(z) \in M_{k/2}(N, \chi_{\Delta^*})$, where $\chi_{\Delta^*}(a) = \left(\frac{\Delta^*}{a}\right)$.

Question

Can we study Λ by studying θ_Λ ? Is $\Lambda \mapsto \theta_\Lambda$ injective?

Isometry and genus

We define the orthogonal group

$$\mathrm{O}(V) = \{g \in \mathrm{GL}(V) : Q(gv) = Q(v)\}$$
$$\mathrm{O}(\Lambda) = \{g \in \mathrm{O}(V) : g\Lambda = \Lambda\}$$

and write $\mathrm{SO}(V)$ and $\mathrm{SO}(\Lambda)$ for those with $\det(g) = 1$. Lattices Λ, Π are **isometric**, written $\Pi \simeq \Lambda$, if there exists $g \in \mathrm{O}(V)$ such that $g\Lambda = \Pi$. The **genus** of Λ is

$$\mathrm{gen}(\Lambda) := \{\Pi \subseteq V : \Lambda_p \simeq \Pi_p \text{ for all } p\}.$$

The **class set** $\mathrm{cls}(\Lambda) = \mathrm{gen}(\Lambda)/\simeq$ is the set of (global) isometry classes in $\mathrm{gen}(\Lambda)$. It is finite, by geometry of numbers.

Non-injectivity of θ

However, even the map $\theta : \mathrm{cls}(\Lambda) \to M_{k/2}(N, \chi_{\Delta^*})$ is not injective.

Example

- $D_k = \{x \in \mathbb{Z}^k : 2 \mid \sum_{i=1}^{k} x_i\}$
- $E_k = D_k + \mathbb{Z} \cdot \frac{1}{2}(1, \ldots, 1)$
- Then $\mathrm{disc}(E_8) = \mathrm{disc}(E_{16}) = 1$, so $\theta_{E_{16}}, \theta_{E_8 \oplus E_8} \in M_8(1)$.
- $\dim M_8(1) = 1$ implies $\theta_{E_{16}} = \theta_{E_8 \oplus E_8}$.
- However, $[E_{16}] \neq [E_8 \oplus E_8]$.

How do we know that?

Theorem (Freitag, 1983)

If $\mathfrak{H}_g = \{z \in M_g(\mathbb{C}) : z^t = z, \Im(z) > 0\}$, *then for* $z \in \mathfrak{H}_g$

$$\theta_{\Lambda}^{(g)}(z) = \sum_{\lambda \in \Lambda^g} e^{\pi i \,\mathrm{Tr}(\lambda^t T \lambda z)} \in M_{k/2}^{(g)}(N, \chi_{\Delta^*})$$

Use $g = 4$.

Neighbors

Kneser's theory of p-neighbors gives an effective method to compute the class set; it also gives a Hecke action!
Let $p \nmid \operatorname{disc}(\Lambda)$ be a prime; $p = 2$ is OK.
We say that a lattice $\Pi \subseteq V$ is a ***p*-neighbor** of Λ, and write $\Pi \sim_p \Lambda$ if

$$[\Lambda : \Lambda \cap \Pi] = [\Pi : \Lambda \cap \Pi] = p.$$

If $\Lambda \sim_p \Pi$ then:

- $\operatorname{disc}(\Lambda) = \operatorname{disc}(\Pi)$,
- Π is integral, and
- $\Pi \in \operatorname{gen}(\Lambda)$.

Moreover, there exists S such that every $[\Pi] \in \operatorname{cls}(\Lambda)$ is an **iterated *S*-neighbor** of Λ.

$$\Lambda \sim_{p_1} \Lambda_1 \sim p_2 \cdots \sim_{p_r} \Lambda_r \simeq \Pi$$

with $p_i \in S$. Typically may take $S = \{p\}$.

Example - Computing the class set

Let $\Lambda = \mathbb{Z}^4$ with the quadratic form

$$Q(x_1, x_2, x_3, x_4) = x_1^2 + x_1x_2 + x_2^2 + x_3^2 + x_1x_4 + x_3x_4 + 3x_4^2$$

and bilinear form given by

$$[T_\Lambda] = \begin{pmatrix} 2 & 1 & 0 & 1 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 1 & 0 & 1 & 6 \end{pmatrix}$$

Thus $\mathrm{disc}(\Lambda) = 29$.

$$\Lambda' = \frac{1}{2}\mathbb{Z}(e_2 + e_4) + 2\mathbb{Z}e_3 + \mathbb{Z}e_1 + \mathbb{Z}e_4$$

with corresponding quadratic form

$$Q(x) = x_1^2 + x_1x_2 + 4x_2^2 + x_1x_3 + x_3^2 + 3x_1x_4 + 2x_2x_4 + x_3x_4 + 3x_4^2$$

Orthogonal modular forms

The space of **orthogonal modular forms** of level Λ is

$$M(\mathrm{O}(\Lambda)) := \{f : \mathrm{cls}(\Lambda) \to \mathbb{C}\} \simeq \mathbb{C}^{h(\Lambda)}$$

For $p \nmid \mathrm{disc}(\Lambda)$ define the **Hecke operator**

$$T_p : M(\mathrm{O}(\Lambda)) \to M(\mathrm{O}(\Lambda))$$
$$f \mapsto \left([\Lambda'] \mapsto \sum_{\Pi' \sim_p \Lambda'} f([\Pi']) \right)$$

The Hecke operators commute and are self-adjoint, hence there is a basis of simultaneous eigenvectors - **eigenforms**. (Gross, 1999)

Example - square discriminant

Let Λ have the Gram matrix

$$[T_\Lambda] = \begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 6 & 0 \\ 1 & 0 & 0 & 6 \end{pmatrix}$$

so that $\mathrm{disc}(\Lambda) = \det T = 11^2$. Then $h(\Lambda) = 3$.
Write $\mathrm{cls}(\Lambda) = \{[\Lambda] = [\Lambda_1], [\Lambda_2], [\Lambda_3]\}$. Then a basis of eigenforms is given by

$$\phi_1 = [\Lambda_1] + [\Lambda_2] + [\Lambda_3], \qquad \phi_2 = 4[\Lambda_1] - 6[\Lambda_2] + 9[\Lambda_3]$$
$$\phi_3 = 4[\Lambda_1] + [\Lambda_2] - 6[\Lambda_3],$$

and we have

$$\theta(\phi_1) = \frac{5}{12} + q + 3q^2 + 4q^3 + 7q^4 + 6q^5 + 12q^6 + O(q^7) \in E_2(11)$$
$$\theta(\phi_2) = q - 2q^2 - q^3 + 2q^4 + q^5 + 2q^6 - 2q^7 + O(q^9) \in S_2(11)$$

where $T_p(\phi_2) = \lambda_p \phi_2$ with $\lambda_2 = 4, \lambda_3 = 1, \lambda_5 = 1, \lambda_7 = 4, \ldots$

Relating the eigenvalues

Theorem (A., Fretwell, Ingalls, Logan, Secord, and Voight (2022), consequence of Rallis (1982))

If k is even, ϕ is an eigenform and $f = \theta^{(g)}(\phi) \neq 0$ with $2g < k$:

$$L(\phi, s) = L\left(\chi_{D^*} \otimes f, \mathit{std}, s - \left(\frac{k}{2} - 1\right)\right) \prod_{i=g-\left(\frac{k}{2}-1\right)}^{\left(\frac{k}{2}-1\right)-g} \zeta\left(s + i - \left(\frac{k}{2} - 1\right)\right).$$

If $g = 1$, then obtain $L(\chi_D \otimes \mathrm{Sym}^2(f), s)$ and zeta factors so

$$\lambda_p = a_p^2 - \chi_{D^*}(p) p^{\frac{k}{2}-1} + p\left(\frac{p^{k-3} - 1}{p - 1}\right)$$

where a_p are the eigenvalues of f.

Example - Nonsquare discriminant

Let Λ be as before with discriminant 29. By checking isometry we compute w.r.t. basis $[\Lambda'], [\Lambda]$

$$[T_2] = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, [T_3] = \begin{pmatrix} 4 & 3 \\ 6 & 7 \end{pmatrix}, [T_5] = \begin{pmatrix} 18 & 9 \\ 18 & 27 \end{pmatrix}, \ldots$$

The constant function $\phi_1 = [\Lambda] + [\Lambda']$ is an **Eisenstein series** with $T_p(\phi_1) = (p^2 + (1 + \chi_{29}(p)) + 1)\phi_1$. Another eigenvector is $\phi_2 = [\Lambda] - 2[\Lambda']$, with $T_p(\phi_2) = \lambda_p \phi_2$

$$\lambda_2 = -1, \lambda_3 = 1, \lambda_5 = 9, \lambda_7 = 4, \lambda_{11} = 17, \ldots$$

But

$$\theta(\phi_2) = q - \frac{3}{2}q^2 + \frac{3}{2}q^3 - 3q^4 - 3q^5 + O(q^6)$$

is not an eigenform. We match them with the **Hilbert modular form** labeled 2.2.29.1-1.1-a in the LMFDB.

Towards a bijection?

Would like to have a bijection between **orthogonal modular forms** and **Hilbert modular forms**, but... Consider $Q(x) = x_1^2 + x_2^2 + x_3^2 + x_1x_4 + x_2x_4 + 3x_4^2$ with Gram matrix

$$[T_\Lambda] = \begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 1 & 1 & 0 & 6 \end{pmatrix}$$

and $\mathrm{disc}(\Lambda) = 40$.

- Then $\dim S(O(\Lambda)) = 1 \neq 2 = \dim S_2(\mathbb{Z}[\sqrt{10}])$.
- This is because of the lattice Λ_2 with form $Q_2(x) = x_1^2 + x_2^2 + 2x^3 + x_2x_4 + 2x_3x_4 + 2x_4^2$.
- Although $\Lambda_2 \notin \mathrm{gen}(\Lambda_1)$, it is everywhere locally **similar** to Λ_1.

Similarity classes

We define the general orthogonal group

$$\begin{aligned} \mathrm{GO}(V) &= \{g \in \mathrm{GL}(V) : Q(gv) = \mu(g)Q(v), \quad \mu(g) \in \mathbb{Q}^\times\} \\ \mathrm{GO}(\Lambda) &= \{g \in \mathrm{GO}(V) : g\Lambda = \Lambda\} \end{aligned}$$

and write $\mathrm{GSO}(V)$ and $\mathrm{GSO}(\Lambda)$ for those with $\det(g) > 0$.
Lattices Λ, Π are **similar**, written $\Pi \sim \Lambda$, if there exists $g \in \mathrm{GO}(V)$ such that $g\Lambda = \Pi$. The **similarity genus** of Λ is

$$\mathrm{sgen}(\Lambda) := \{\Pi \subseteq V : \Lambda_p \sim \Pi_p \text{ for all } p\}.$$

The **similarity class set** $\mathrm{scls}(\Lambda) = \mathrm{sgen}(\Lambda)/\sim$ is the set of (global) similarity classes in $\mathrm{sgen}(\Lambda)$. It is finite, by geometry of numbers.

Main Theorem

Assume that $k = 4$.

Theorem (A., Fretwell, Ingalls, Logan, Secord, and Voight (2024))

Assume $\mathrm{disc}(\Lambda) = D_0 N^2$, $K = \mathbb{Q}[\sqrt{D_0}]$. *Then*

$$S(GO(\Lambda)) \hookrightarrow S_2(N\mathbb{Z}_K)_{G_K}$$

with image the projection of $S_2(N\mathbb{Z}_K; W = \epsilon)^{D\text{-new}}$

- $G_K = \mathrm{Gal}(K|\mathbb{Q})$ acts naturally on the space of Hilbert modular forms.
- D is the product of the anisotropic primes.
- For $p \mid N$, we set $\epsilon_p = -1$ if $p \mid D$, else $\epsilon_p = 1$.
- W_p is the Atkin-Lehner involution at $p\mathbb{Z}_K \mid N\mathbb{Z}_K$.

More generally...

- Works over a totally real field F.
- Functorial, hence arbitrary weight.
- Twisting by the spinor norm, can obtain other AL signs.
- Can identify explicitly the space $M(O(\Lambda))$ using twisting by Hecke characters and unit characters.
- Lattices have to be **residually binary everywhere**.

Key ideas - Quaternions and even Clifford

The even Clifford algebra $B = C_0(V)$ is quaternion with center K.
It also maps lattices in V to orders in B.
Even Clifford extends to a map

$$C_0 : \mathrm{GSO}(V) \to (B^\times \times F^\times)/K^\times.$$

Theorem (A., Fretwell, Ingalls, Logan, Secord, and Voight (2024))

The even Clifford map induces an isomorphism

$$C_0^* : M_\rho(C_0(\Lambda)^\times, \psi^{-1} \circ \mathrm{Nm}_{K|F})^{AL_F(C_0(\Lambda))} \longrightarrow M_{C_0^*\rho}(\mathrm{GSO}(\Lambda), \psi).$$

- Sends $\mathfrak{P}$-neighbors to $\mathfrak{P}$-neighbors.
- Sends $\mathfrak{p}^1$-neighbors to $\mathfrak{p}\mathbb{Z}_K$-neighbors.
- Also induces $C_0 : \mathrm{GO}(V)/F^\times \to \mathrm{Aut}_F(B)$, with

$$0 \to B^\times/K^\times \simeq \mathrm{Aut}_K(B) \to \mathrm{Aut}_F(B) \to \mathrm{Gal}(K|F) \to 0.$$

$\bullet\ \bullet \cong$ $\bullet\ \bullet$ [$A_1 \times A_1 = D_2$, equiv. $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2 \cong \mathfrak{so}_4$]

Applications - non vanishing

We obtain commutative diagrams of Hecke modules

$$
\begin{array}{ccc}
S(\mathcal{O})^{AL_F(\mathcal{O})}_{G_K} & \xleftrightarrow{\;\;C_0^*\;\;} & S(\mathrm{GO}(\Lambda)) \\
\Big\updownarrow{\scriptstyle JL} & & \Big\downarrow{\scriptstyle \theta_\Lambda^{(2)}} \\
S(\mathfrak{N}\mathbb{Z}_K, W = \epsilon)^{\mathfrak{M}\text{-new}}_{G_K} & \longrightarrow & S^{(2)}(\Gamma_0^{(2)}(\mathfrak{N}), \chi_K)
\end{array}
$$

The bottom line is:

- Yoshida lift when $K = F \times F$ and f, g are both cuspidal.
- Saito-Kurokawa lift when $K = F \times F$ otherwise.
- Asai lift when K is a field.

Shows that when K is a field (e.g. $D = 4p$ (Kaylor, 2019)), θ_2 is injective (non-vanishing).

Applications - Hilbert modular surfaces

Corollary ((Chan, 1999), Theorem 1)

Let $K = \mathbb{Q}(\sqrt{D})$, and let Y_ϵ be the components of the Hilbert modular surface over K.

$$\sum_{\epsilon \in \mathrm{Cl}^+(K)/\mathrm{Cl}^+(K)^2} (p_g(Y_\epsilon) + 1) = \sum_{\Lambda} \# \operatorname{cls}(\mathrm{SO}(\Lambda)),$$

where Λ ranges over genera of lattices of discriminant D.

Proof.

Main theorem implies $\dim_{\mathbb{Q}} S_2(\mathbb{Z}_K) = \dim_{\mathbb{Q}} S(\mathrm{GSO}(\Lambda))$. But

$$\# \mathrm{Cl}^+(K)/\mathrm{Cl}^+(K)^2 = \#\mu(\operatorname{cls}(\mathrm{GSO}(\Lambda))) = \dim_{\mathbb{Q}} E(\mathrm{GSO}(\Lambda)),$$

$$\dim_{\mathbb{Q}} S_2(\mathbb{Z}_K) + \# \mathrm{Cl}^+(K)/\mathrm{Cl}^+(K)^2 = \dim_{\mathbb{Q}} M(\mathrm{GSO}(\Lambda)),$$

which yields the result. □

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